THE THEORY OF MIXTURES

—A System of Mathematics Invented by an Amateur—

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PREFACE

When we have to make a calculation to find the concentration or composition ratios of a certain kind of solution or mixture, a knowledge of elementary arithmetics would be enough, but if a problem is more complicated, we cannot help depending upon algebra for its solution, setting up some equations with two or three unknowns that need careful thinking accompanied by a tedious calculation.

Suppose we have to solve such problems as shown below:

Example 1. We have certain quantities of 12–carat, 18–carat, and 24–carat gold. If we are going to obtain 125g of 22–carat gold by adding certain amounts of the first two grades of gold to 100g of the 24–carat (or pure) gold, how many grams each would have to be added?

Example 2. Here is 100g of 20% saline solution (salt water). To have 2.0g of pure solid salt deposited at 25°C by evaporation, how much water would have to be removed, provided the concentration of saturated salt water at 25°C is 26.6%?

Although these problems do not involve any particular principles that are hard to understand, they could be bewildering and sometimes too tedious to find their correct answers promptly.

For young students to train their brains by solving complicated mathematical problems may be significant, but to the average people the arithmetic or algebraic thinking of this kind would be nothing but the waste of time and brains. What is essential is to find the right answers as easily and quickly as possible.

According to the Theory of Mixtures, the answers to the above examples are very easily found to be 16.67g and 8.33g, and 30.3g, respectively.

Apart from their logical grounds, we are going to illustrate briefly how those answers are to be obtained.

The solution of Example 1 is carried out in three steps:

Step 1. Translate the statement into the 'mixture equation'

$$100[24]+x[12]+y[18] = 125[22]$$

(where $100+x+y = 125$)

Step 2. Subtract 12 from each number in the bracket, thus

$$100[12]+x[0]+y[6] = 125[10]$$

Step 3. Multiply each number in the bracket by each co-efficient, add up all the products on either side, and equate them.

$$100\times12+x\times0+y\times6 = 125\times10$$

Hence $6y = 50, \quad \therefore y = \underline{8.33}(\text{g})$

Since $100+x+y = 125, \quad x = \underline{16.67}(g)$

The solution of Example 2 is also much the same.

The statement of the problem is translated into the equation

$$100[20]-x[0] = 2[100]+y[26.6]$$

where $100-x = 2+y$

(Note that the symbol [0] stands for 0% salt water, i.e., pure water, and [100] for pure salt)

In this case, calculation is even simpler because the second term $-x[0]$ makes subtraction unnecessary. Thus we have in the same way

$$100\times20 = 2\times100+26.6y$$

$$y = \frac{1800}{26.6}$$

$$= 67.67$$

Since $100-x = 2+y$

$$x = 100-2-67.67$$

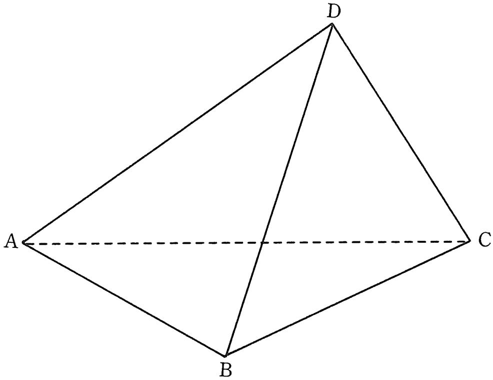
$$= \underline{30.33}(g)$$

The writer once asked a mathematics teacher to have a group of his high school students solve this same question, with the result that almost all of them gave wrong answers due to the wrong equations they had set up. I still wonder "What if they had learned The Theory of Mixtures then?"

In the last chapter we shall discuss the application of the present theory to geometry, the underlying idea of which is outlined as follows:

We plot four points (origins) A, B, C, and D in space. If we let each of these four origins stand for a particular pure substance, then

(1) Any point on each line connecting two origins represents a mixture composed of two of the four pure substances.



(2) Any point on each plane determined by three origines represents a mixture composed of the three of the four pure substances.

(3) Any point in space other than the above represents a mixture composed of the four pure substances.

The use of composition ratios of mixtures as co-ordinates has turned out to be capable of solving very easily some of the problems which seem to be tedious to solve by dint of a usual geometrical method.

This paper is a presentation of the Theory of Mixtures, a system of mathematics invented by the writer himself who is an utter amateur in mathematics.

Needless to say, the theory is perhaps very trivial mathematically, but the writer would be contented, if it should help encourage anyone who is going to venture on a creative work, whatever kind of work it may be.

Chapter 1　Composition Formulas

1.1　Expression of Mixtures

When we say, "Here are three different substances A, B, and C," the letters A, B, or C do specify the species of the individual substances, but not their amounts whatever. In the present theory, we will let them express not only the species but also the amounts of substances, thus for example, let 1 unit, 2 units, or 3 units of these substances be expressed by A, 2B, or 3C, respectively.

Next, we will consider the mixture of substances. When, for example, a grams of A, b grams of B, and c grams of C are mixed together, we will describe this by the formula

$$aA+bB+cC$$

in which the addition sign (+) means the process or operation of 'mixing' besides the meaning of the addition in algebra.

As for the units of quantity, we will exclusively use the units of weight or mass for the present, because some substances may be accompanied by a change in volume when mixed.

Now, the mixture thus formed is composed of three components A, B, and C in the ratios a: b: c, with its total amount a+b+c=m. We will describe this mixture as $mA_aB_bC_c$, and call the formula a **composition formula.** Let us call m the **quantity co-efficient** (or solely the coefficient) of the composition formula, and a, b, and c each the **ratio index** for the corresponding component.

1.2 The Basic Equation for Mixture

If we use the equality sign (=) for the equality of both the species and the amount of substances, we can obtain the equation

$$\begin{aligned} aA+bB+cC &= mA_aB_bC_c \\ &= mA_{a'}B_{b'}C_{c'} \qquad [1.1] \end{aligned}$$

where $a+b+c = m$

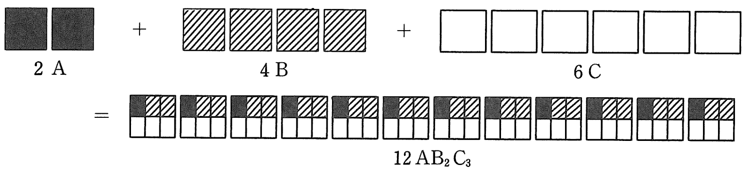
$a: b: c = a' : b' : c'$

This equation is the very foundation of the Theory of Mixtures, on which all the subsequent discussion is based; so we will name the equation the **Basic Equation for Mixture** in particular.

Example 1.1 When 2g of a substance A, 4*g* of B, and 6*g* of C are mixed together to form an even mixture, we have

$$2A+4B+6C = 12A_2B_4C_6$$
$$= 12AB_2C_3$$

The following illustration may help comprehend the meaning of the example:



1.3 Resolution, Addition, and Subtraction of Composition Formulas

From the definition of the composition formula, which is expressed as the above Basic Equation for Mixture, it is clear that any composition formula expressed by $mA_aB_bC_c$ can be resolved into components as follows:

$$mA_aB_bC_c = \frac{ma}{a+b+c}A + \frac{mb}{a+b+c}B + \frac{mc}{a+b+c} \qquad [1.2]$$

(This relation can also be obtained by dividing each co-efficient in the Basic Equation by a+b+c.)

Example 1.2 $100A_2B_3C_5 = \frac{100\times2}{2+3+5}A + \frac{100\times3}{2+3+5}B + \frac{100\times5}{2+3+5}C$

$$= 20A+30B+50C$$

The addition of mixtures, that is, the mixing of two or more different mixtures to form a new mixture, can be expressed by the addition of composition formulas. The formula for the new mixture can be determined by first resolving each given composition formula into components, and then reunifying all the resulting components, as shown by the following example.

Example 1.3 Calculate $100A_2B_3C_5+200A_5B_4C$

Solution.

$$\begin{aligned} 100A_2B_3C_5 &= 20A+30B+50C \\ 200A_5B_4C &= 100A+80B+20C \\ 300ABC_3 &= \underline{60A+60B+180C} \\ & \quad 180A+170B+250C = 600A_{18}B_{17}C_{25} \end{aligned}$$

Subtraction can be done in the same manner, but care must be taken when negative numbers appear among the ratio indexes of a composition formula. See the example that follnws:

$$\begin{aligned} & 200A_5B_4C-100A_2B_3C_5 \\ &= 100A+80B+20C-(20A+30B+50C) \\ &= 80A+50B-30C \\ &= 100A_8B_5C_{-3} \end{aligned}$$

No mixture whose composition formula has negative ratio indexes can exist in actuality, though it doesn't matter at all in the course of calculation. We will call such a formula with negative ratio indexes a **negative composition formula,** distinguishing it from an ordinary (positive) one.

1.4 The meaning of Negative Composition Formulas

Negative composition formulas do not appear very often in practical calculation, but since in the last chapter dealing with a graphic expression of composition formulas they will prove to play a very important role, it would be better to examine them briefly at this point.

For simplification, we will examine the negative composition formula A_2B_{-1} as a sample.

The formula A_2B_{-1} is resolved into components in accordance with the rule, thus

$$A_2B_{-1} = 1\times\frac{2}{2-1}A+1\times\frac{-1}{2-1}B$$
$$= 2A-B$$

Adding B to both sides(or transposing B), we have

$$A_2B_{-1}+B = 2A$$

The equation states that a unit amount of this strange mixture denoted by A_2B_{-1} would become a double amount of the pure substance A, if another unit of a foreign substance B were to be added to it. It is, as it were 'too pure by itself.'

To make the matter more understandable, we are going to compare three mixtures A_2B, AB_0, and A_2B_{-1} with respect to the contents of its component A, thus:

for A_2B, the ratio of A to tolal $=\frac{2}{2+1}=\frac{2}{3}$ (66.7% A)

for AB_0, the ratio of A to tolal $=\frac{1}{1+0}=1$ (100% A)

for A_2B_{-1}, the ratio of A to tolal $=\frac{2}{2-1}=2$ (200% A)

Why, the mixture A_2B_{-1} is 200% pure, too pure to exist in actuality!

1.5 Deformation of a Composition Formula

From the Basic Equation for Mixture are derived some important rules for the deformation of a composition formula. For convenience' sake, we will discuss the subject, using composition formulas with a limited number of components.

Rule 1. Any composition formula remains equivalent if each of its ratio indexes is multiplied by the same number except 0, thus, for example,

$$mA_aB_bC_c = mA_{ak}B_{bk}C_{ck} \quad (k \neq 0)$$

This is evident from the definition of the composition formula itself.

Example 1.4 Make sure that the rule is true even when k is a negative number, for example, $A_2B_{-1}=A_{-2}B$.

$$A_2B_{-1} = \frac{2}{2-1}A + \frac{-1}{2-1}B$$
$$= 2A-B$$
$$A_{-2}B = \frac{-2}{-2+1}A + \frac{1}{-2+1}B$$
$$= 2A-B$$

Therefore $\quad A_2B_{-1} = A_{-2}B$

Note: That a composition formula cannot have its ratio indexes multiplied by 0 does not mean that it cannot have zero among its ratio indexes. (For example, $2A+0B=2A_2B_0=2A$. Any composition formula whose coefficient is 0 is equivalent to nothing; the formula itself can be erased in this case)

Rule 2. If in a composition formula $A_aB_bC_cD_d$ an arbitrary combination of any two or more components, A_aB_b for example, is regarded as a particular substance and denoted by (A_aB_b), we have

$$A_aB_bC_cD_d = (A_aB_b)_{a+b}C_cD_d \qquad [1-3]$$

The converse is also true.

The rule may alse be expressed as follows:

Let $A_aB_b = X$, then $A_aB_bC_cD_d = X_{a+b}C_cD_d$

The rule can easily be proved true as follows:

$$(a+b+c+d)\ A_aB_bC_cD_d = aA+bB+cC+dD$$
$$= (a+b)\ A_aB_b+cC+dD$$
$$= (a+b+c+d)\ (A_aB_b)_{a+b}C_cD_d$$

Example 1.5 (1) $A_2B_3C_7 = (A_2B_3)_5C_7$

(2)　$(A_2B_3)_5(CD)_2 = A_2B_3CD$

Rule 3. Any component in a composition formula can be resolved into parts, with the sum of ratio indexes unchanged, thus for example

$$A_aB_bC_c = A_{a_1}A_{a_2}B_{b_1}B_{b_2}C_{c_1}C_{c_2} = \text{etc.} \qquad [1-4]$$

in which　$a = a_1+a_2$, $b = b_1+b_2$, $c = c_1+c_2$, etc.

The converse is also true.

Example 1.6　(1)　$A_3B_2C_5 = AA_2B_2C_2C_3$

$$= A_3B_3B_{-1}C_6D_{-1} = \text{etc.}$$

(2)　Make sure　$A = AA = AAA$

Solution　$AA = \frac{1}{2}A+\frac{1}{2}A = A$

$$AAA = \frac{1}{3}A+\frac{1}{3}A+\frac{1}{3}A = A$$

Example 1.7　Calculate $2ABC+A_2B$ by using the above mentioned rules.

Solution.　$2\,ABC+A_2B = 3\,(ABC)_2(A_2B)$

$= 3\,(A_2B_2C_2)_6(A_2B)_3$ …… Rule 1

$= 3A_2B_2C_2A_2B$ …… Rule 2

$= 3A_4B_3C_2$ …… Rule 3

1.6　Application of the Deformation Rules

All that has been discussed on the deformation of composition formulas seems self-evident and even boring, but in time it will turn out to be very effective in solving some practical problems, especially in chemistry and geometry. In this chapter we are going to refer to two chemical problems to illustrate the use of the deformation of composition formulas.

Problem 1.1　When equal amounts by weight of hydrogen and oxygen are mixed and reacted to form water, calculate the ratio of the amount of water to that of a remaining gas. (Tell what the gas is.)

Solution. The chemical formula H_2O shows the ratio by weight of hydrogen to oxygen to be 2: 16, so that a unit amount of water can be expressed by the composition formula, HO_8. Therefore, we have

$$H_8O_8 = HO_8H_7 \qquad \text{(Rule 3)}$$

$$= (HO_8)_9H_7 \qquad \text{(Rule 2)}$$

Thus we know immediately that the remaining gas must be hydrogen and that the ratio of the water to the gas is 9: 7 by weight.

Problem 1.2 The solubility of boric acid per 100g of water is 5.0g at 20°C, and 23.6g at 80°C. When 100g of saturated boric acid solution at 80°C is cooled to 20°C, how many grams of crystalline boric acid would be deposited?

Solution. Using B for boric acid, and W for water, the saturated boric acid solutions at 80°C and 20°C are expressed by the composition formulas $B_{23.6}W_{100}$ and $B_5\ W_{100}$, respectively. Therafore we have

$$100\ B_{23.6}W_{100} = 100\ B_5B_{18.6}W_{100} \qquad \text{(Rule 3)}$$

$$= 100\ (B_5W_{100})_{105}\underline{B_{18.6}} \qquad \text{(Rule 2)}$$

The underlined part indicating the amount of boric acid to be deposited, we find it to be

$$100\times\frac{18.6}{105+18.6} = \underline{15.05}(g)$$

The solution mentioned alove is so simple that one may think the problem itself is also very simple, but the fact is no. For comparison, we are going to see several other familiar ways to solve the problem.

The first is the one quotated from the chemistry reference book for high school students in which this same problem is found.

The solution given by the author of the book reads as follows:

"The largest amount of boric acid that is soluble in 100g of water at 80°C is 23.6g, while it is only 5.0g at 20°C. Therefore, if the saturated boric acid solution is cooled to 20°C, 23.6−5.0=18.6g of boric acid should be separated from the solution. This amount is, however, that which would be deposited from 100+23.6=123.6g of the solution. So the amount to be deposited from 100g should be

$$18.6\times\frac{100}{123.6}=\underline{15.05}(\mathrm{g})$$

The arithmetical reasoning mentioned above, although corresponding, in fact, to a deformation of the composition formula, is far from being understandable.

Another solution using algebra might be as follows:

Let x(g) be the amount of boric acid to be deposited, and then the amount of boric acid contained in the solution at 20°C after deposition should be $(100-x)\times\frac{5}{100+5}$ g. Equating the total amounts of boric acid before and after deposition, we have the eguation

$$100\times\frac{23.6}{100+23.6}=x+(100-x)\times\frac{5}{100+5}$$

Rearranging,

$$\frac{100}{105}x=\frac{2360}{123.6}-\frac{500}{105}$$

$$x=\frac{105}{100}\left(\frac{2360}{123.6}-\frac{500}{105}\right)=\underline{15.05}(\mathrm{g})$$

Lastly we are going to show one of the wrong solutions often found among students' answers, as follows:

The amount of boric acid contained in 100g of the saturated boric acid solution at 80°C is $100\times\frac{23.6}{100+23.6}$=19.10g, while that which is contained in the same amount of saturated solution at 20°C must be $100\times\frac{5}{100+5}$ =4.75g. The difference between the two, 19.10−4.75=14.35g, should therefore be deposited.

The solution is wrong because deposition from a saturated solution is accompanied by a decrease in the total amount of the solution itself, and so the actual amount of the solution at 20°C should be less than 100g.

Chapter 2 The Concentration-type of Composition Formulas

In this chapter we will introduce another form or type of composition formulas more convenient for use in practical calculations.

2.1 The Concentration–type of Composition Formulas

When, in a certain mixture, the ratio of one of its components, A to the total amount of the mixture is equal to c, we will describe a unit amount of this mixture as A[c], and call this type of formula a **concentration type composition formula,** and the number n its **concentration index.**

As to a composition formula $A_a B_b C_c$, for example, the ratio of one of its components, A to the total being $\frac{a}{a+b+c}$, it can be expressed as $A[\frac{a}{a+b+c}]$. In the same way, the formula may also be expressed as $B[\frac{b}{a+b+c}]$ or $C[\frac{c}{a+b+c}]$ with respect to the other components B or C.

Transformation from the concentration-type composition formula A[c] to the **standard type** (Let us call so) is easy: Since the ratio of any one component A to the rest is c: 1−c, using the symbol R for the rest, we have

$$A[c] = A_c R_{1-c}$$
$$= cA+(1-c)R \qquad [1-5]$$

As is evident from its definition, the value for the concentration index c should be normally

$0 < c \leqq 1$

Four cases are considered as to the values for c:

ⅰ. If $c = 1$, $A[1] = AR_0$

$$= A + 0R = A, \text{ thus}$$

A[1] represents a unit amount of a pure substance A.

ⅱ. If $c = 0$, $A[0] = A_0R$

$$= 0A + R = R, \text{ thus}$$

A[0] represents a unit amount of a substance containing none of the substance A.

ⅲ. If $c > 1$, for example, $c = 1.2$,

$$A[1.2] = A_{1.2}R_{1-1.2}$$
$$= A_{1.2}R_{-0.2}$$

The formula is a negative composition formula mentioned in the previous chapter.

ⅳ. If $c < 0$, for example, $c = -2$

$$A[-2] = A_{-2}R_{1-(-2)}$$
$$= A_{-2}R_3$$

This is also a negative composition formula.

Example 2.1

(1) $A_2B_3C_5 = A\left[\frac{2}{2+3+5}\right] = A[0.2]$

(2) $A[0.3] = A_{0.3}R_{1-0.3} = A_3R_7$

2.2 Change of Components

If a mixture composed of two components A and B is expressed in the form of A[c], it follows that

$$A[c] = A_cB_{1-c} = B[1-c] \qquad [6]$$

If we consider one of the two components as the **solute** and the other as its **solvent**, we might say that Formula [6] is the formula for interchange of solute and solvent.

We may as well refer briefly to the definition of the chemical terms 'solute' and 'solvent' at this point, for we shall have to discuss them again in the subsequent chapters.

When, for exanple, a certain amount of salt is dissolved in water to form a solution, chemistry tells, salt is the solute, and water its solvent. If, however, both components are liquid, the designations solute and solvent are more ambiguous. For instance, in a solution containing 1g of alcohol in 100g of water, alcohol may well be called the solute, and water its solvent, but on the contrary, if only 1g of water is dissolved in 100g of alcohol, nobody would speak of water as its solvent. The chemist suggests that the component present in the greater amount should be called the solvent.

Example 2.2 An aqueous solution containing alcohol 30% can be expressed in two ways, using A for alcohol and W for water, thus

$$A_3W_7 = A[0.3]$$

$$\text{or} \quad A_3W_7 = W[1-0.3] = W[0.7]$$

Such denotations again may seem insignificant, but it is helpful, especially in dealing with the so called **conjugate solution,** a solution composed of two components which are soluble in each other, exemplified by ether/water or phenol/water. When a pair of such liquids are mixed, we obtain a solution in which two saturated solutions composed of both components are present in two phases. Calculation of the amounts of these saturated solutions is sure to be very complicated but it can be done quite easily all the same by means of the Theory of Mixtures. The illustration of concrete examples will be discussed in detail in the next chapter.

2.3 The Addition Rule

The addition of concentration-type composition formulas with the same component A can be done exclusively by means of the following formula, which we will call the **addition rule.**

$$m_1A[c_1]+m_2A[c_2]+\cdots = (m_1+m_2+\cdots)A\left[\frac{m_1c_1+m_2c_2+\cdots}{m_1+m_2+\cdots}\right] \qquad [7]$$

The proof is given as follows:

$$m_1A[c_1] = m_1A_{c_1}R_{1-c_1} = m_1c_1A+m_1(1-c_1)R$$

$$m_2A[c_2] = m_2A_{c_2}R_{1-c_2} = m_2c_2A+m_2(1-c_2)R$$

$$\vdots \qquad\qquad \vdots \qquad (+$$

$$m_1A[c_1]+m_2A[c_2]+\cdots = (m_1c_1+m_2c_2+\cdots)A$$

$$+\ (m_1+m_2+\cdots-m_1c_1-m_2c_2-\cdots)R$$

$$= (m_1+m_2+\cdots)Am_1c_1+m_2c_2+\cdots$$

$$Rm_1+m_2+\cdots-m_1c_1-m_2c_2-\cdots$$

$$= (m_1+m_2+\cdots)A\left[\frac{m_1c_1+m_2c_2+\cdots}{m_1+m_2+\cdots}\right]$$

Example 2.3 $20A[0.1]+30A[0.2]+50A[0.3]$

$$= (20+30+50)A\left[\frac{20\times0.1+30\times0.2+50\times0.3}{20+30+50}\right]$$

$$= 100A[0.23]$$

Chapter 3 Mixture Equations

We have hitherto seen many equations consisting of composition formulas, but they have all been the equations which belong in the so called ‘identical equatins’ in algebra which hold under any condition. Those equations of composition formulas that can hold under a certain specific condition we will call **mixture equations.**

We have two types of mixture equations depending on the type of the composition formulas involved, the standard-type and the concentration-type.

3.1 The Standard–type of Mixture Equations

If we limit the species of components to A, B, and C, any standard-type mixture equation can be expressed in the following general form:

$$m_1 A_{a_1} B_{b_1} C_{c_1} + m_2 A_{a_2} B_{b_2} C_{c_2} + \cdots$$
$$= m_1' A_{a_1'} B_{b_1'} C_{c_1'} + m_2' A_{a_2'} B_{b_2'} C_{c_2'} + \cdots \qquad [8]$$

where $\quad m_1 + m_2 + \cdots = m_1' + m_2' + \cdots$

If there are any unknowns in a given equation, we should be able to find their numerical values according to the conditions given. Some examples are shown below.

Example 3.1 We have two mixtures, both composed of A and B. One is a mixture with the composition ratio of A to B=2: 3, the ratio of the other unknown. Assuming that the two mixtures are mixed in the ratio of 5: 2 to give a mixture with the composition ratio 1: 1, calculate the composition ratio of the latter mixture.

Solution. The statement of the question may be expressed by (or translated into) the equation

$$5A_2B_3 + 2A_xB_y = 7AB$$

Transposing the first term, we have

$$\begin{aligned} 2A_xB_y &= 7AB - 5A_2B_3 \\ &= 3.5A + 3.5B - (2A + 3B) \\ &= 1.5A + 0.5B \\ &= 2A_{1.5}B_{0.5} \\ &= 2A_3B \end{aligned}$$

Thus, we know the ratio of A to B of the latter mixture must be 3:1.

There are some similar problems that require a slight operational technique, as illustrated by the following example.

Example 3.2 We have three alloys. One is 20g of gold (20%) plus silver (80%), and the others are 30g of silver/copper, and 50g of copper/gold, the composition ratios for the latter two being unknown. Assuming that these three alloys were mixed together to give an alloy of 39% gold, 28% silver and 33% copper, colculate the composition ratios of the two alloys.

Solution. If we use A, B, and C for gold, silver and copper, respectively, the three alloys are designated respectively as $20A_2B_8$, $30B_xC_y$, and $50C_{y'}A_{x'}$. In accordance with the statement, we have

$$20A_2B_8+30B_xC_y+50C_{y'}A_{x'} = 100A_{39}B_{28}C_{33}$$

Transposing the first term,

$$30B_xC_y+50C_{y'}A_{x'} = 100A_{39}B_{28}C_{33}-20A_2B_8$$

$$= 39A+28B+33C-4A-16B$$

Rearranging

$$= 12B+33C+35A$$

Here, taking the coefficients of both sides into account, we divide 33C into 18C and 15C, thus

$$= \underbrace{12B + 18C}\, + \underbrace{15C + 35A}$$

$$= 30B_{12}C_{18}+50C_{15}A_{35}$$

$$= 30B_2C_3+50C_3A_7$$

$$= 30B_xC_y+50C_{y'}A_{x'}$$

Therefore we find the compositions of the two alloys must be silver: copper=2:3, and copper: gold=3:7.

Although the standard-type mixture equations can show very clearly the species of substance, the compositions, and the relatiove quantities of the mixtures involved, they frequently fail to be effective in a practical calculation, as will be seen from the following example:

Example 3.3 Solve the equation

$$xAB_2+yA_3B = 100AB$$

where $$x+y = 100 \qquad \cdots\cdots\cdots (1)$$

The meaning of the equation is very clear, but its solution is rather tedious.

Solution. Resolving into components

$$\frac{x}{3}A+\frac{2x}{3}B+\frac{3y}{4}A+\frac{y}{4}B$$

$$=\frac{4x+9y}{12}A+\frac{8x+3y}{12}B = 100AB$$

Therefore we have

$$4x+9y = 8x+3y$$

Hence $$4x-6y = 0 \qquad \cdots\cdots\cdots (2)$$

By solving the simultaneous equations (1) and (2), we abtain $x=60$ and $y=40$.

However, the calculation is too tedious to meet our demand.

Instead of using standard-type mixture equations, which seem a little too formal for practical calculations, we had better employ the concentration-type equations which would be handier and more effective for the purpose.

3.2 The concentration–type of Mixture Equations

The concentration-type of mixture equations with the same component A can be expressed as the general equatin

$$m_1A[c_1]+m_2A[c_2]+\cdots = m_1'A[c_1']+m_2'A[c_2']+\cdots \qquad [9]$$

where $$m_1+m_2+\cdots = m_1'+m_2'+\cdots$$

From this general formula, we can derive several very important laws as follows:

Law 1. The sum of the products of coefficient and concentration index on either side is equal to each other, thus

$$m_1c_1+m_2c_2+\cdots = m_1'c_1'+m_2'c_2'+\cdots$$

Proof. Applying the addition rule to both sides of Equation [9]

$$(m_1+m_2+\cdots)A\left[\frac{m_1c_1+m_2c_2+\cdots}{m_1+m_2+\cdots}\right] = (m_1'+m_2'+\cdots)A\left[\frac{m_1'c_1'+m_2'c_2'+\cdots}{m_1'+m_2'+\cdots}\right]$$

Since $m_1+m_2+\cdots = m_1'+m_2'+\cdots$, it follows that

$$m_1c_1+m_2c_2+\cdots = m_1'c_1'+m_2'c_2'+\cdots$$

Example. 3.3 In the mixture equation

$xA[0.5]+yA[0.1] = 100[0.3]$, we have

$$x\times0.5+y\times0.1 = 100\times0.3$$

$$5x+y = 300$$

Law 2. A concentration-type mixture equation remains valid if the same number is added to, or subtracted from, each of the concentration indexes.

For simplification, we are going to show that if the following Equation (1) holds, Equation (2) also holds, where n is an arbitrary number.

$$xA[c_1]+yA[c_2] = x'A[c_1']+y'A[c_2'] \quad \cdots\cdots\cdots (1)$$

$$xA[c_1-n]+yA[c_2-n] = x'A[c_1'-n]+y'A[c_2'-n] \quad \cdots\cdots\cdots (2)$$

Applying the addition rule to Equation (2),

$$xA[c_1-n]+yA[c_2-n] = (x+y)A\left[\frac{x(c_1-n)+y(c_2-n)}{x+y}\right]$$

$$= (x+y)A\left[\frac{c_1x+c_2y-n(x+y)}{x+y}\right]$$

Since $x+y = x'+y'$

$$c_1x+c_2y = c_1'x'+c_2'y' \quad \text{(Law 2)}$$

Therefore the left side is equal to

$$(x'+y')A\left[\frac{c_1'x'+c_2'-n(x'+y')}{x'+y'}\right]$$

$$= (x'+y')A\left[\frac{x'(c'_1-n)+y'(c'_2-n)}{x'+y'}\right]$$

$$= x'A[c'_1-n]+y'A[c'_2-n] = \text{the right side}$$

Example 3.4 Find the values for x and y from the equation

$$xA[0.5]+yA[0.1] = 100[0.3]$$

Solution. Subtracting 0.1 from each concentration index,

$$xA[0.4]+yA[0] = 100[0.2]$$

Applying Law 1,

$$0.4x = 20$$

$$\therefore \quad x = 50$$

Since $x+y = 100$, we have

$$\underline{x = 50} \qquad \underline{y = 50}$$

Law 3. A concentration-type mixture equation remains valid if each of the concentration indexes is multiplied, or divided, by the same number. This is also true of the coefficients.

The law is so probable that we will omit its demonstration, which can be done in a similar manner. Instead, we must mention another notation of a composition formula at this point.

As we have seen, the above mentioned equation inExample 3.4

$$xA[0.5]+yA[0.1] = 100[0.3]$$

is equivalent to the equation

$$xA[5]+yA[1] = 100[3]$$

in which each concentration index has been multiplied by the same number 10.

According to our definition, however, the notation A[5], for example, should stand for an imaginary mixture, because

$$A[5] = A_5R_{1-5} = A_5R_{-4}$$

This does not matter mathematically, of course, but in dealing with practical problems this type of notation can bring about a confusion.

For instance, a mixture of x grams of 24-carat gold and y grams of 18-carat gold can be expressed in the formula $x\mathrm{A}\left[\frac{24}{24}\right]+y\mathrm{A}\left[\frac{18}{18}\right]$, but not by $x\mathrm{A}[24]+y\mathrm{A}[18]$, the latter being admitted only in the form of a mixture equation, as, for example,

$$x\mathrm{A}[24]+y\mathrm{A}[18] = 100\mathrm{A}[20]$$

In order to avoid such inconvenience, we will employ another notation, like $x[24]$ or 100[20], with the general form m[n] lacking in the capital letter symbolic of the component. Thus, we might say that the composition formula m[n] represents m units (by weight) of a mixture whose relative concentration or density as compared with the other like mixtures involved is indicated by the number n; we will call n an **index number.**

3.3 Application of Concentration-type Mixture Equations to Practical Problems

The three laws mentioned above are very efficient in the solution of concentration-type mixture equations. We are going to solve several practical problems by dint of these laws. Most of the problems to be dealt with are the ones which have already been presented as examples for illustration but to which no complete solutions have been given yet.

Problem 3.1 To make 100g of 15.0% aqueous alcohol solution by mixing 10.5% and 20.5% ones, how many grams each of the two solutions are to be mixed?

Solution. Let x and y be the amounts to be found, then we have the mixture equation

$$x[10.5]+y[20.5] = 100[15.0]$$

Subtracting 10.5 from each index number

$$x[0]+y[10] = 100[4.5]$$

Applying Law 1, we have

$$10y = 450 \qquad \therefore \quad \underline{y = 45}(\mathrm{g})$$

Since the equation shows $x+y = 100$, $\underline{x = 55}(\mathrm{g})$

Problem 3.2 We have certain quantities of 12-carat, 18-carat and 24-carat gold. If we are going to obtain 125g of 22-carat gold by adding certain amounts of the first two grades of gold to 100g of the 24-carat (or pure) gold, how many grams each of them would have to be added?

Solution. (This is the same problem presented as an example at the top of this paper.)

In accordance with the statement, we have

$$100[24]+x[18]+y[12] = 125[22]$$

Subtracting 12 from each index number

$$100[12]+x[6]+y[0] = 125[10]$$

Appling Law 1

$$1200+6x = 1250 \qquad \underline{x = 8.33}(\mathrm{g})$$

Since the equation shows $100+x+y = 125$

$$y = 125-100-8.33 = \underline{16.67}(\mathrm{g})$$

Note. Average students would solve this same problem by means of algebraic equations as follows:

Solution. Considering the amounts of pure gold contained in the individual grades of gold, we have the equation

$$100\times\frac{24}{24}+\frac{18}{24}x+\frac{12}{24}y = 125\times\frac{22}{24}$$

Simplifying

$$18x+12y = 350 \qquad \cdots\cdots\cdots (1)$$

while $x+y = 25$ .... (2)

Solution of these simultaneous equations gives the answers $\underline{x=8.33}$(g), $\underline{y=16.67}$(g).

The problem is so simple that one may think there is no great difference in easiness between the two methods, but look at the next problem.

Problem 3.3 Here is 100g of 20% saline solution (salt water). To have 2.0g of solid salt deposited at 25°C by evaporation, how much water should be removed, provided that the concentration of saturated salt water at 25°C is 26.6%? (Note that salt can be dissolved in water until its concentration reaches 26.6%, over which the solution can contain no more salt, the excess deposited as crystal.)

Solution. If we denote 20% salt water by [20], pure salt should be denoted by [100], and pure water by [0]. Let x(g) be the amount of water to be removed, and y(g), that of the saturated salt water present after deposition, then we can obtain the following mixture equation in accordance with the statement.

$$100[20]-x[0] = 2[100]+y[26.6]$$

Applying Law 1 $\quad 2000 = 200+26.6y$

Hence $\quad y=\dfrac{1800}{26.6}=67.67$

Since $\quad 100-x = 2+y$

$$x = 100-2-67.67 = \underline{30.33}(\text{g})$$

So it turns out that 30.33g of water should be removed.

Now, let us try to solve the problem by means of a usual algebraic method. One way of thinking might be as follows:

Solution. Let x(g) be the amount of water to be removed, then the total amount of saturated salt water will be $(98-x)$g because the given solution (100g) will lose xg of water in evaporation, and 2g of salt in deposition. On the other hand, the amount of salt contained in the saturated salt water will be 18g because the initial salt content $100\times0.2=20$g decreases by 2g through deposition. Thus, the ratio of the salt present in the saturated salt water to its total amount must be $\frac{18}{98-x}$, which should be equal to 0.266 (26.6%). Therefore, we have the equation

$$\frac{18}{98-x}=0.266$$

Rearranging $98-x=\frac{18}{0.266}$

Hence $x=98-67.67=\underline{30.33}(\text{g})$

Lastly, we must refer to another example involving the 'conjugate solution', a problem which was already presented as an example but has been reserved to be solved in this chapter.

Problem 3.4 Water and ether (ethyl ether) can dissolve each other, with the sobubility of water to ether at 20°C being 98.76 (percentage by weight) and that of ether to water, 6.87.

(1) Write a mixture equation that shows 100g of water and 5g of ether are mixed to form two saturated solutions.

(2) Calculate the amounts of these solutions.

Solution. Using E for ether, W for water, and x and y for the unknown amounts, we have the equation

$$100\text{W}+5\text{E}=x\text{W}[0.986]+y\text{E}[0.0687]$$

This is the mixture equation required. The above equation is a very clear expression of the statement, but it needs transforming for its solution:

$$100\text{W}+5\text{E}=105\text{W}_{100}\text{E}_5$$

$$=105\text{E}\left[\frac{5}{105}\right]$$

$$=105\text{E}[0.0476]$$

$$=105[4.76]\quad(\times100)$$

$$W[0.9876] = E[1-0.9876]$$
$$= E[0.0124]$$
$$= [1.24] \quad (\times 100)$$
$$E[0.0687] = [6.87] \quad (\times 100)$$

Thus, the equation becomes

$$105[4.76] = x[1.24]+y[6.87]$$

Subtracting 1.24 from each index number, we have

$$105[3.52] = x[0]+y[5.63]$$

By Law 1 $\quad 105\times 3.52 = 5.63y$

$$y = \frac{105\times 3.52}{5.63}$$
$$= \underline{65.6}(g)$$

Since the equation shows $x+y=105$

$$x = 105-65.6$$
$$= \underline{39.4}(g)$$

Chapter 4 Application of Composition Formulas to Geometry

The idea of the mixture theory enables us to solve some sorts of problems in geometry. This is done by expressing the position of any point on a plane or in space as a composition formula for a mixture.

4.1 Expression of Points on a Line by Composition Formulas

Let us consider a segment $\overline{AB}$ on which lies an arbitrary point P diving the segment in a ratio a: b as shown in Fig. 1, and let one end of the segment, A stand for a pure substance A, and the other end B, another kind of pure substance B, then it would be reasonable to define that the point P should represent a mixture denoted by $A_a B_b$.

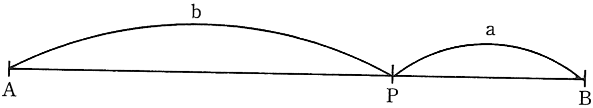


Fig. 1

It must be noted that the point P should be designated as $A_a B_b$, not as $A_b B_a$, because, as seen from the figure, the nearer the point to B, the smaller the relative amount of A. This is a usual expresession referred to as the 'lever relation'.

Now, so for as point P lies between A and B just as shown in Fig. 1, it is representative of a real mixture of A and B. However, if it shifts to the right beyond the terminal point B (Fig.2), or inversely beyond A to the left (Fig.3), the composition formula for the point becomes a negative one, thus for example:

in Fig.2, $A_a B_b = A_{-2} B_5$

$$= A\left[\frac{-2}{3}\right] = B\left[\frac{5}{3}\right]$$

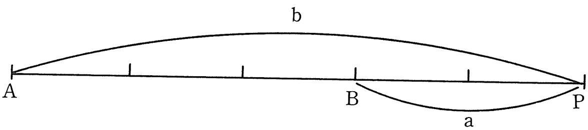


Fig. 2

in Fig.3, $A_a B_b = A_5 B_{-2}$

$$= A\left[\frac{5}{3}\right] = B\left[\frac{-2}{3}\right]$$

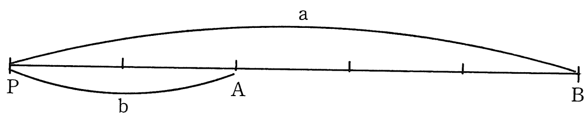


Fig. 3

If we employ the notation A_xB or A_xB_{1-x} instead of A_aB_b, with x for a variable, it follows that the composition formula A_xB or A_xB_{1-x} indicates every point on the line that connects the two points A and B; in other words, the line $\overline{AB}$ can be expressed by a two-component composition formula with a single independent variable in its ratio indexes.

4.2 Expression of Points on a Plane by Composition Formulas

In order to express a point on a plane as a composition formula, we need, in addition to the above mentioned segment $\overline{AB}$, another origin point C which does not lie on the segment and which stands for another species of pure substance, as shown in Fig.4. For simplification, let us illustrate how a point on the plane determined by the three points A, B, and C is expressed by a composition formula.

In Fig.4, any point lying on the line $\overline{AB}$ represents a mixture of A and B. (Point D, for example, is expressed as the composition formula A_2B_3 according to lever relation.) Now, when we connect D with C, we may consider that any point on the line $\overline{CD}$ should express a particular mixture of C and A_2B_3, its composition ratio indicated by its relative position

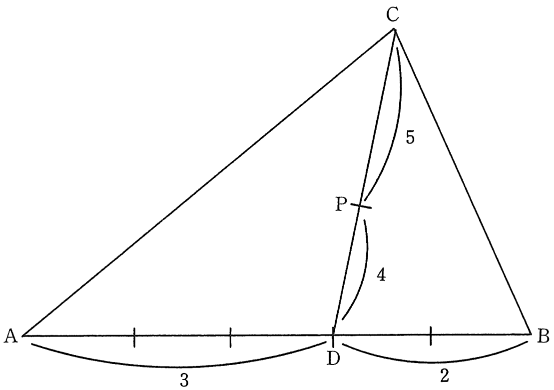


Fig. 4

on the segment. Thus, if we plot on the line $\overline{CD}$ a point P that divides the segment internally in the ratio of, say, $\overline{CP}$: $\overline{PD}$=5:4, the point P is expressed by the composition formula $(A_2B_3)_5C_4=A_2B_3C_4$. On the other hand, when we connect A with C, and B with C, then any point on the segments $\overline{AC}$ and $\overline{BC}$ indicates, respectively, a mixture of A and C, and of B and C, just as any point on the segment $\overline{AB}$ indicates a mixtures of A and B.

To sum up, we might conclude that any point inside or outside the triangle ABC can be expressed by a general formula $A_aB_bC_c$, depending on a, b, and c being all positive or not.

Furthermore, if we employ a notation, such as AB_xC_y, $A_xB_yC_{1-x-y}$, etc. with x and y for independent variables, we can have it express any point on the plane determined by the three points A, B, and C.

Example 4–1. In the parallelogram ABCD, show the vertex D by a composition formula.

Solution. The intersection of the two diagonals, E is expressed as E=AC. Since the segment $\overline{BE}$ is divided externally by the point D in the ratio 2:1, the composition formula for D should be

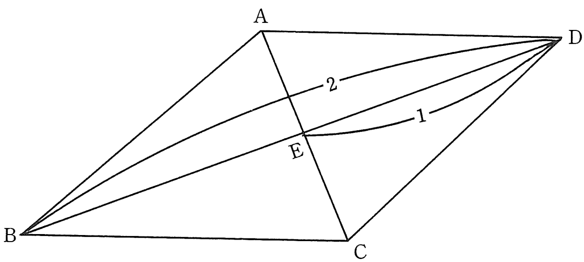


Fig. 5

$$D = E_2B_{-1}$$
$$= (AC)_2B_{-1}$$
$$= \mathbf{AB_{-1}C}$$

4.3 The Intersection of Two Lines

The intersection of two lines can easily be expressed by a composition formula, as illustrated by the following example.

Example 4.2 In the triangle ABC in Fig.6, Point D bisects the side AC, and Point E trisects the side $\overline{AB}$. Show the composition formula for the intersection of the two lines $\overline{BD}$ and $\overline{CE}$.

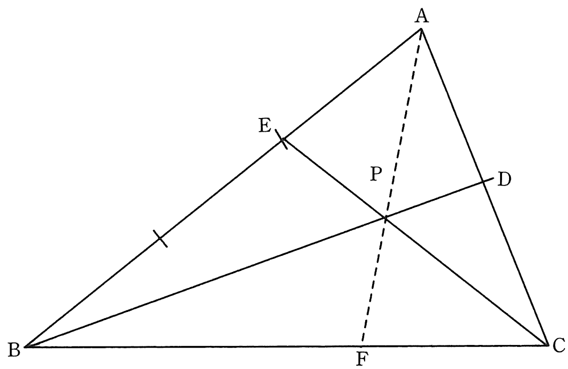


Fig. 6

Solution. Since D=AC and E=A_2B, the line $\overline{BD}$ is expressed as $B_x(AC)$, and the line $\overline{CE}$ as $C_y(A_2B)$. Since the two formulas must be identical at the intersection, we should have

$$B_x(AC) = C_y(A_2B)$$

Deforming both sides so as to make the indexes for A be equal,

$$B_x(AC) = B_{4x}(AC)_4 = B_{4x}(A_2C_2)_4 = A_2B_{4x}C_2$$
$$C_y(A_2B) = C_{3y}(A_2B)_3$$
$$= A_2BC_{3y}$$

The intersection is therefore indicated by

$$A_2B_{4x}C_2 = A_2BC_{3y} = \underline{A_2BC_2}$$

Note. See how a slight deformation of the formula for the intersection provides important information as shown below:

$$A_2BC_2 = A_2(BC_2)_3 \text{ indicates } \overline{BF}:\overline{FC} = 2:1$$
$$\text{and } \overline{AP}:\overline{PF} = 3:2$$
$$A_2BC_2 = (A_2C_2)_4B \text{ indicates } \overline{BP}:\overline{PD} = 4:1$$
$$A_2BC_2 = (A_2B)_3C_2 \text{ indicates } \overline{CP}:\overline{PE} = 3:2$$

4.4 Solution of Geometrical Problems by Composition Formulas

To show the efficacy of the present theory we are going to solve some typical problems including a couple of well-known theorems in geometry by using composition formulas.

Problems 4.1 Show that the three medians of a triangle meet at one point, dividing each median in the ratio of 1: 2. (The theorem for the center of gravity)

Solution. In the triangle ABC (Fig.7), the midpoint of one side, D, and that of another side, E are expressed by the composition formulas BC and AC, respectively, and therefore the two medians can be expressed as $A_x(BC)$ and $C_y(AB)$, respectively. So the intersection P should be

$$P = A_x(BC) = C_y(AB)$$

Since $$A_x(BC) = A_xB_{\frac{1}{2}}C_{\frac{1}{2}}$$
$$C_y(AB) = A_{\frac{1}{2}}B_{\frac{1}{2}}C_y$$

Therefore $$x = \frac{1}{2},\ y = \frac{1}{2}$$

So the intersection must be expressed by ABC. The formula $ABC=(AB)_2C$ evidently shows that the line $\overline{CP}$ is identical with the third median of the

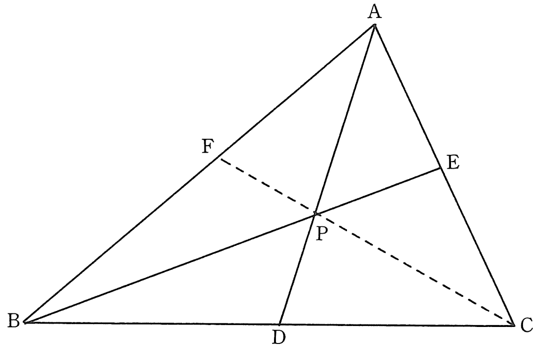


Fig. 7

triangle, demonstrating that the three medians meet at Point P. The formula for the intersection, ABC(=A(BC)$_2$=B(CA)$_2$) shows at the same time that it divides each median in the ratio 1:2.

Problem 4.2 Show that the three bisectors of a triangle meet at a single point. (Theorem for the incenter)

Solution. Let two bisectors of a triangle ABC be $\overline{AD}$ and $\overline{BE}$, and the length of each side be equal to a, b, and c as shown in Fig.8. $\overline{AD}$ and $\overline{BE}$ being bisectors, it is known that

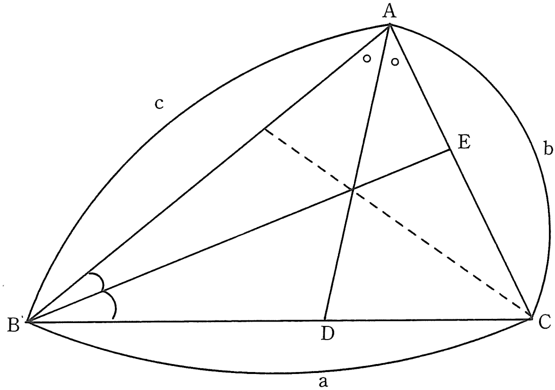


Fig. 8

$\overline{BD}:\overline{DC} = c:b$

$\overline{AE}:\overline{EC} = c:a$

Therefore $D = B_b C_c$

$E = C_c A_a$

Since $\overline{AD}$ may be represented by $A_x B_b C_c$, and $\overline{BE}$ by $B_y C_c A_a$, the intersection P is determined by $P = A_x B_b C_c = B_y C_c A_a$

Thus $P = A_a B_b C_c$

In the same way, the intersection of another pair of bisectors P′ is also found to be $A_a B_b C_c$, which is to verify that the three bisectors of a triangle meet at the same point $A_a B_b C_c$.

Problem 4.3 In the parallelogram OABC, Point D divides one side $\overline{AB}$ internally in the ratio 2:3, and Point E divides internally the diagonal $\overline{AC}$ in the ratio 2:5.

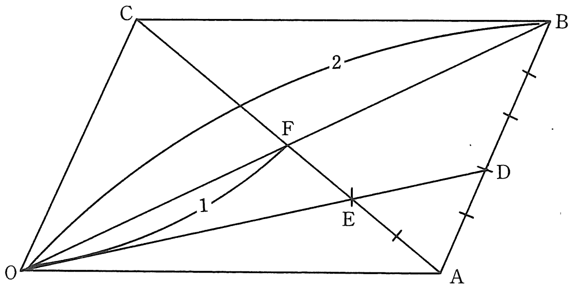


Fig. 9

Show that the three points O, E, and D are on a straight line.

Solution. Let the intersection of the two diagonals be F, then F=AC. Since Point O divides the segment $\overline{BF}$ externally in the ratio 2:1,

$$O = B_{-1}(CA)_2 = AB_{-1}C \quad \text{(Rule 2)}$$

On the other hand, the line $\overline{OE}$ can be expressed as

$$\overline{OE} = (AB_{-1}C)_x A_5 C_2 \qquad \text{(Note 1−1+1=1)}$$
$$= A_x B_{-x} C_x A_5 C_2 \qquad \text{(Rule 2)}$$
$$= A_{x+5} B_{-x} C_{x+2} \qquad \text{(Rule 3)}$$

The line $\overline{OE}$ should meet the line $\overline{AB}$ at a point where the index ratio for C is equal to 0, that is, where $x+2=0$ or $x=-2$.

Therefore the formula for the intersection must be

$$A_{-2+5} B_{-(-2)} C_0 = A_3 B_2 \ (=D)$$

Thus, the line $\overline{OE}$ turns out to meet the line $\overline{AB}$ at a point identical with Point D, which shows that O, E, and D lie on the same line.

4.5 Expression of Points in Space by Composition Formulas

As would be expected from the above discussion, any point in space can be expressed by a four-component mixture. Another origin D is needed

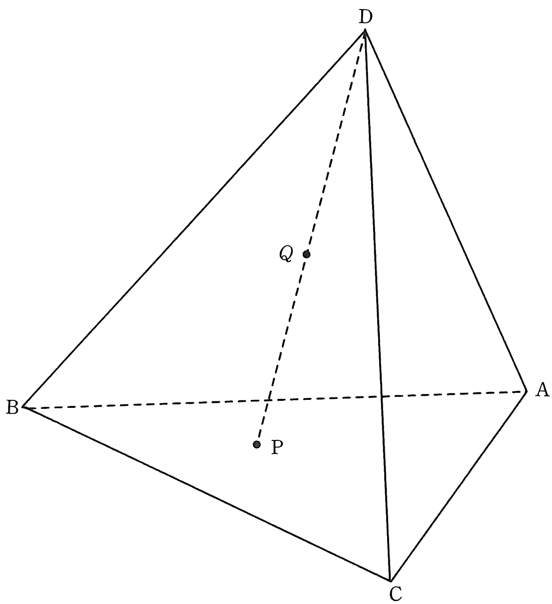


Fig. 10

which lies over or under the plane ABC, with four points forming a triangular pyramid or a tetrahedron as shown in Fig.10. To sum up, the meaning of the tetrahedron in Fig.10 should be defined as follows:

Ⅰ. All that has been true of the triangle ABC is also true of the other three triangles DAB, DBC, and DCA, that is to say,

(1) Each vertex, A,B,C, or D indicates a particular pure substance.

(2) Any point between two vertexes on each edge represents a particular binary mixture.

(3) Any point on each surface rapresents a particular three-component mixture.

Ⅱ. Any point in space inside or outside the tetrahedron represents a four-component mixture expressed by a general formula $A_a B_b C_c D_d$, depending on all the ratio indexes being positive or not.

Ⅲ. The co-ordinates, or the composition formula for any point (Q in Fig.10) is determined as follows: Firstly the vertex D is connected with the point Q, and the line $\overline{DQ}$ is extended to meet the opposite surface at P. Then the position of Q on the line $\overline{DP}$ relative to D and P will give its co-ordinates in the form of a mixture consisting of the pure substance D and a three-component mixture of A, B and C.

Example 4.3 In the co-oridinate system ABCD shown in Fig.11, E is a point on the line $\overline{DC}$, and F is a point on the plane ABC. Provided that the points E and F are expressed by C_2D and AB_2C_3, respectively, determine the composition formula for a point G at which the line $\overline{EF}$ meets the opposite plane ABD.

Solution. The line to connect E with F is expressed by

$$E_x F = (C_2D)_x (AB_2C_3)^*$$

$$= C_{\frac{2x}{3}} D_{\frac{x}{3}} A_{\frac{1}{6}} B_{\frac{2}{6}} C_{\frac{3}{6}}$$

$$= C_{4x} D_{2x} AB_2C_3$$

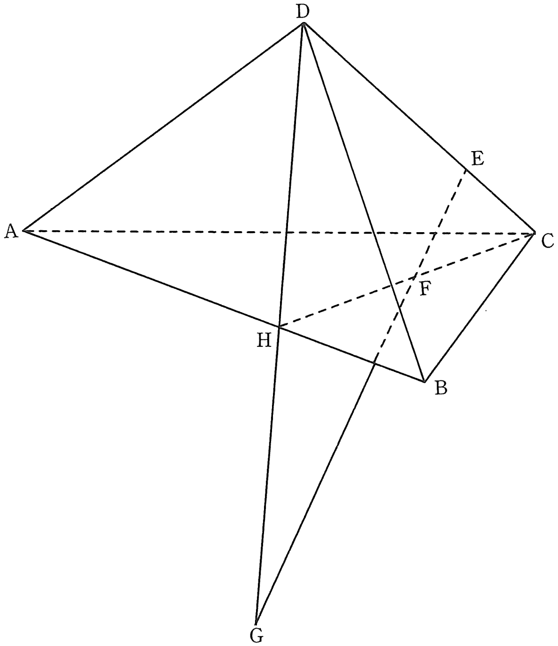


Fig. 11

$$= AB_2C_{4x+3}D_{2x}$$

Since the intersection of the line $\overline{EF}$ and the plane ABD contains none of the component C, the ratio index for C being 0, we have

$$4x+3 = 0, \qquad x = -\frac{3}{4}$$

Therefore, the intersection G should be

$$G = AB_2C_0D_{2\times(-\frac{3}{4})}$$
$$= AB_2D_{-\frac{3}{2}}$$
$$= A_2B_4D_{-3}$$
$$= (A_2B_4)_6D_{-3}$$
$$= (AB_2)_2D_{-1}$$

If we let H indicate the intersection of the lines $\overline{DG}$ and $\overline{AB}$,

$$G = H_2D_{-1}$$

which shows that Point G divides the segment $\overline{DH}$ ($H=AB_2$) externally in the ratio 2:1.

* **Note.** The formula for the line $\overline{EF}$ is more conveniently expressed as $(C_2D)_{3x}AB_2C_3=C_{2x}D_xAB_2C_3=AB_2C_{2x+3}D_x$.

The result is the same, of course, thus

From $2x+3=0$, $x=-\dfrac{3}{2}$. Therefore $G = AB_2D_{-\frac{3}{2}} = A_2B_4D_{-3}$

4.6 Solution of Problems in Space Geometry by Means of Composition Formulas

As another example for the geometric application of four-component composition formulas, let us solve a problem in space geometry as the last problem in this paper.

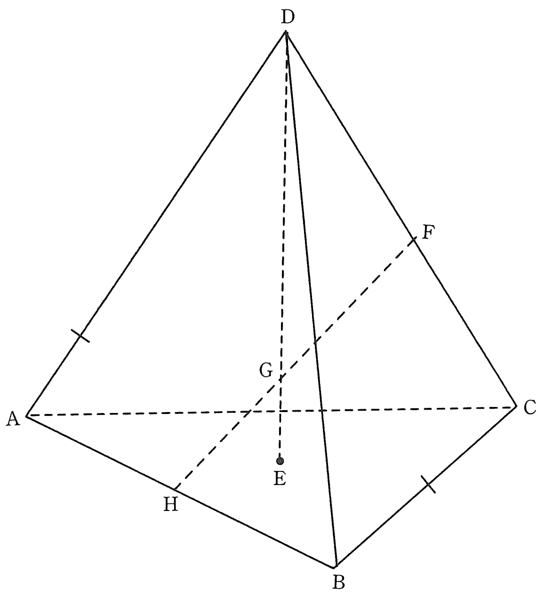


Fig. 12

Problem 4.4 In a tetrahedron ABCD in Fig.12, G is the point that divides internally in the ratio 3: 1 the segment which connects a vertex D with the center of gravity of its opposite surface, E. Show that Point G is identical with the midpoint of the segment which connects the midpoints of a pair of opposite edges (two edges not lying on the same plane of a tetrahedron).

Solution. First, we determine the composition formula for E, which is found to be ABC from the equation

$$A_x(BC) = B_y(CA) \quad \text{(cf. Chap 4–4.4)}$$

Point G is, therefore, expressed by

$$(ABC)_3D = ABCD$$

Since the midpoints of the segments $\overline{AB}$ and $\overline{CD}$ are expressed by the formulas AB and CD, respectively, the midpoint of the segment connecting the said two midpoints should be

$$(AB)(CD) = ABCD = G$$

This proves that Point G is the midpoint of the segment connecting the midpoints of a pair of opposite edges of a tetrahedron.

The End